mission: mars

One: Introduction

Research into orbital mechanics traditionally takes place along two paths. The first seeks an analytical solution, using pure mathematics to model the system of forces. The second path uses computer-based numerical solution methods to solve a system of equations. Many practical problems in the motion of bodies reach a point beyond which there is no analytical solution, at which time it is necessary to resort to numerical methods. The topic of this report, the minimum energy Hohmann Transfer, is such a problem. The analysis begins with the basics, then subjects the exact mathematical solution to a comprehensive numerical analysis.

Just as an experimentalist devotes much effort toward setting up his or her apparatus before actually beginning observations, so too with the researcher into celestial mechanics. The traditional scientific researcher is sure to use the most dependable and accurate laboratory apparatus. In our research into the Hohmann Transfer, we must have the best equipment as well, be it in the form of equations, integrators, or high precision digital computers. This equipment generates our data. We want it to be as accurate as possible, just as the chemist or physicist strives to obtain the very latest mass spectrograph or accelerometer. Once we conduct our experiments, we will analyze our results in the same fashion as any other scientific experimentalist.

There are those who would rush to judgment and skip the analytical part of the investigation altogether, going instead directly to the numerical problem. This is especially tempting, because computers are so powerful and computer time so inexpensive. There are, however, important benefits to first building an understanding of the elementary principles. It only ensures a thorough understanding of the system under investigation, but aids in the interpretation of the data generated by our laboratory apparatus.

ASSUMPTIONS

The trajectory of a Hohmann Transfer is between two orbits that have a common central body. Figure 1.1 illustrates a typical configuration. The path of the spacecraft (s/c) is half of an ellipse with the sun at one focus. The extremes of the ellipse, the apoapse and periapse, correspond to the points of intersection with the initial and final orbits. The ideal Hohmann Transfer maneuver is a simple realignment of an orbit to another orbit around the same central body. A communication satellite going from a low earth orbit (LEO) to a geosynchronous orbit (GEO) is one application.

<<Figure 1.1>> The Hohmann Transfer

The traditional Hohmann trajectory begins with a thrust, projecting the s/c into an elliptical transfer orbit. A second thrust at apoapse re-configures the orbit at the new altitude. Both thrusts are tangential to the flight path (the flight path angle (FPA) being zero at apoapse and periapse), thus all of the energy is used in changing the shape of the orbit. Consequently, the Hohmann Transfer is called a minimum energy trajectory. (The bi-elliptic trajectory can sometimes use less energy, but the time in transit is far greater than a Hohmann Transfer, and the incremental energy savings is nominal.)

The assumptions made in this paper concerning the Hohmann Transfer are as follows:

1.The thrusts are instantaneous

2. All the orbits are in the same plane

3. The initial and final orbits are circular

These are the usual constraints on the Hohmann Transfer.

This paper will investigate a specific application, a transfer from one body in orbit around a central body to a second body in orbit around the same central body. The Mars Pathfinder mission followed such a trajectory in its flight from Earth to Mars. Such a path differs from the usual Hohmann Transfer because the s/c leaves one body and arrives at a second body, in an orbit that requires close coordination with the two orbits. In this sense, it is more appropriately termed a Hohmann Rendezvous. The above constraints still apply:

1. The thrusts are instantaneous

2. The orbits of the two bodies and the s/c are in the same plane

3. The two bodies are in circular orbits

A final assumption is that the mass of the s/c is negligible in comparison to the other three bodies. This model is illustrated in Figure 1.2.

<<Figure 1.2>> The Hohmann Rendezvous

The coplanar Hohmann Rendezvous has significant differences to the transfer, primarily due to the affect of the source and target bodies upon the s/c. These will be investigated in detail, first analytically then numerically, in the following sections. In any event, the above assumptions will remain valid for the entire study.

Two: The Analytical Solution

This chapter presents the fundamental development of the Hohmann Transfer. This leads naturally to the introduction of some relevant principles of mission analysis. These ideas, such as the sphere of influence and the patched conic method, are important in the study of the Hohmann Transfer as a practical means of executing an interplanetary rendezvous.

The minimum energy Hohmann Transfer has a very specific geometry (see Figure 1.2). If the initial and final orbits of the s/c are circular, with radii of r_1 and r_2, respectively; then the major axis of the transfer orbit is

$2a = r_1 + r_2$ (2.1)

The eccentricity of the elliptical transfer orbit is also a function of r_1 and r_2,

$e = (r_2 - r_1)/\ r_1 + r_2$ (2.2)

as is the semi-parameter of the transfer orbit

$2/p = 1/r_1 + 1/r_2$ (2.3)

These equations assume a common central body for all three orbits: the initial, final, and transfer orbit.

Referring again to Figure 1.2, notice that the apoapse is at r_1 and the periapse of the transfer orbit is at r_2, and the s/c travels exactly half of the ellipse. Two tangential burns are required, one at r_a and one at r_p, the apoapse and periapse, respectively.

The first thrust initiates the s/c flight onto the transfer ellipse, from its circular parking orbit. The velocity of the s/c in this circular parking orbit is

$v_{circ} = \mathrm{sqrt}\ (\mu\ /\ r_p)$ (2.4)

where r_p is the radius of the circular orbit, or r_1 and µ is the gravitational parameter of the central body. The target velocity is the periapse velocity of the transfer orbit.

$v_{circ} = \mathrm{sqrt}(\mu\ /\ r_p) * (1 + e)$ (2.5)

where, in the special case of the circular orbit, $r_p = r_1$.

Combining the expressions (2.4) and (2.5), and the thrust is

¤ $v_1 = \mathrm{sqrt}(\mu\ /\ r_1)\ [\ (1 + e) - 1]$ (2.6)

A similar expression can be developed for the delta-v (¤ v) needed at the apoapse of the transfer orbit.

$v_a = \mathrm{sqrt}\ [(\mu\ /\ r_p)\ (1 + e)/(1 - e)]$ (2.7)

which, subtracting from the circular velocity needed to maintain an orbit at radius r_1 gives the delta-v needed

¤ $v_2 = \mathrm{sqrt}(2\mu\ /\ r_2) - \mathrm{sqrt}(\mu\ /\ r_a)\ [(1 + e)/(1 - e)]$ (2.8)

Both thrusts are assumed to be instantaneous and tangential to the velocity vector. The only orbital element affected, then, is the eccentricity of the s/c orbit.

RENDEZVOUS

The analysis so far is for a spacecraft under the influence of a single central force. An example is a s/c going from a low earth orbit (LEO) at an altitude of 400 km to a circular geosynchronous earth orbit (GEO) at 20,000 km. It is, in this regard, a classical two body problem and it can be solved directly in terms of all six orbital elements.

Another application of the Hohmann Transfer considers the trajectory when there is an object at one or both thrust points, i.e. at apoapse and periapse of the transfer orbit. The space shuttle in LEO might launch a satellite into the elliptical transfer orbit, with the satellite performing the thrust maneuver at periapse. A more complex problem is for the shuttle in LEO to rendezvous with a satellite in a higher orbit. This introduces the timing, or phasing, problem into the analysis. Maintaining the configuration used so far, with the shuttle in a lower orbit than its target, the geometry is shown in Figure 2.1.

<<Figure 2.1>> Coplanar Hohmann Rendezvous

This is still a maneuver between circular orbits, with the bodies all remaining in the same plane throughout the maneuver. As indicated, the target leads the shuttle by a phase angle when the thrust is applied. The object in the lower orbit has a greater mean motion, so the target must lead the shuttle by this phase angle, ø .

$ø = w_{tgt} T_{trans} - pi = pi\ (n_{tgt}\ sqrt(a^3_{trans}/\mu) - 1)$ (2.9)

where w_{tgt} is the angular velocity of the target and n_{tgt} is the mean motion of the target, both in radians/sec. Keep in mind that no mass has been assumed for either the shuttle or the target. Consequently it is still a two body problem.

CONJUNCTION

Two bodies are in conjunction if a straight line from the central body outward intersects both bodies at the same time. Figure 2.2 shows the earth-mars trajectory with the s/c on a Hohmann Transfer path.

<<Figure 2.2>> Earth-Mars Conjunction

Notice the lead angle for Mars. Since the Earth has a greater mean motion it will pass Mars during the time of flight, so the two planets will be in conjunction roughly half way through the s/c trajectory, as indicated.

It is less obvious that there will be two more conjunctions: (1) between earth and the s/c because the initial velocity of the s/c is greater than earth's mean motion, and (2) between mars and the s/c later in the flight path profile. Another way to perceive these three conjunctions is shown in Figure 2.3. This is a plot of time versus phase angle.

<<Figure 2.3>> Three Conjunctions in Perspective

The two planets have a constant angular velocity so they move along straight lines. The Earth's slope is greater than Mars', and the s/c is shown to follow a sinusoidal path between Earth at launch time and Mars at arrival time. The three conjunctions are shown as intersections between the three curves, one for each body. The time when Earth and Mars are in conjunction, or the y-axis in Figure 2.3, is called the syzygy.

SPHERE OF INFLUENCE

The analysis will now concentrate on the Earth-Mars Hohmann rendezvous in greater detail. As the s/c is launched from Earth it is immediately subject to forces, primarily emanating from the Earth and the sun. Initially, the gravitational influence of the Earth is far greater than that of the sun, which at this point is only a third body perturbation. The Hohmann Transfer, however, is predicated on the assumption that the sun is the central force throughout the interplanetary maneuver.

The concept of a s/c being variously under the influence of more than one central force during the course of its trajectory is embodied in the ideal of sphere of influence. Analytically, there is a point on the flight path at which the influence of (in our case) the Earth and the sun are equal. A value for this distance from the smaller body is given by the function

$r_{soi} = (m_{planet}/m_{sun})^{2/5}\ r_{planet}$ (2.10)

This radius divides the s/c trajectory into two regimes: (1) the Earth as central force, and (2) the sun as the central force. The Hohmann Transfer is only conceptually accurate in the latter case.

The two body analysis can still be applied within the sphere of influence by studying the s/c trajectory with respect to the Earth (with the sun "turned off" or considered only as a third body perturbation). Whereas the s/c trajectory beyond SOI is an ellipse, within SOI it is an hyperbolic escape trajectory. Each system must be studied individually, resulting in a final flight path for the s/c that is a "patched conic."

HYPERBOLIC ESCAPE

Looking at the near Earth aspect of the transfer orbit, the s/c is assumed to be in a circular parking orbit. As before, a single tangential thrust is applied at the periapse of what will become an elliptical heliocentric transfer path. On the near Earth scale, however, the thrust must be calculated differently. The semi-major axis is calculated as before (eqn. 2.1), so the energy of the heliocentric transfer is

$E_H = - \mu_{sun}/2a$ (2.11)

where the velocity on the transfer ellipse is

$v^+ = \text{sqrt}\ \mu_{sun}/(2/r_{sun} - 1/a_H)$ (2.12)

Now, the s/c must use the orbital velocity of Earth, v_E, advantageously. This velocity is given by

$v_E = \text{sqrt}\ \mu_{sun}/r_E$ (2.13)

Comparing these relative velocities is best done in a diagram.

<<Figure 2.4>> Heliocentric Relative Velocities

Since the periapse velocity is greater than the velocity of Earth, the difference must be made up by the hyperbolic escape velocity of the s/c. Thus, in vector notation

$| v^{+B\ B}{}_{/E} | + | v_E | = | v^+ |$ (2.14)

Now, to determine the conditions by which this hyperbolic excess velocity can be achieved. The geometry of the situation is given in Figure 2.5.

<<Figure 2.5>> Hyperbolic Escape Trajectory

Noticing that the energy of the escape trajectory is

$E = v^{2B\ B}{}_{/E} / 2 = v_{p/E} / 2 - \mu_E / r_{p/E}$ (2.15)

The required periapse velocity will then be

$v_{p/E} = \text{sqrt}\ (v^{2B\ B}{}_{/E} + 2\ \mu_E / r_{p/E})$ (2.16)

This, then, is the total velocity required of the s/c to put it on the proper escape trajectory. The thrust required is the difference between $v_{p/E}$ and the velocity of the s/c in its circular parking orbit

¤ $v = v_{p/E} - \text{sqrt}\ (\mu_E / r_{p/E})$ (2.17)

This thrust will put the s/c on an escape path that at a great distance will have the $v^{+B\ B}{}_{/E}$, or in the heliocentric frame, the v_p to enter the elliptic Hohmann Transfer ellipse.

<<Figure 2.6>> Transfer Trajectory SOI's

Figure 2.6 gives an overall perspective of the problem, indicating that the vB B of the s/c is the v_p for the Hohmann Transfer.

COMPUTER MODELING

It will be shown in the next chapters that the numerical integration of the interplanetary transfer has much the same problem as the patched conic method, and must also be resolved using two different models. The above analytical presentation, however, lets the Hohmann Transfer be investigated as an isolated problem in orbital mechanics. It will be assumed that the s/c can be put on a range of possible trajectories so that a specific flight path angle and velocity occur at the SOI. Since each of these sub-SOI trajectories can be done at the same cost in terms of thrust (see Figure 2.7), it will only be necessary to determine the optimal flight path angle and velocity at SOI to reach mars at the proper place in its path.

<<Figure 2.7>> Heliocentric Trajectories

The second half of the optimization will be to determine the minimum thrust to get the s/c from a parking orbit around Earth so that when it reaches the SOI it has the proper flight characteristics to reach Mars as planned by the previous analysis. Ideally there will be no thrust required at this point, although it is more likely that a small correction will be applied at Earth SOI. This is still within the minimum energy constraint of the three-thrust orbital transfer trajectory. In fact, given the convenient proximity of the moon's orbit to the SOI staging point, it is possible that a fly-by of the moon will provide this in-flight correction without any expenditure of thrust. All of these possibilities will be investigated in Chapter 4.

Three: The Numerical Solution

A numerical model of the Hohmann Transfer was developed so that the forces from all three principal bodies on the s/c could be integrated simultaneously. The basic equation used is Newton's second law developed into an inertial force equation using the Universal Law of Gravitation.

$F_{g/sat} = Gm_{sun}(r / r^3)$ (3.1)

where the mass of the s/c is neglected.

The motions of earth and mars are not integrated by this method. Since they are assumed to be in circular orbits, they move with a constant mean motion

$n_E = \text{sqrt} (\mu_E + \mu_{sun}) / r^3$ (3.2)

which gives a mean orbital angular velocity, from which the position of Earth and Mars can be determined at any instant by

$x = r_E \cos (n t)$

$y = r_E \sin (n t)$ (3.3)

The position of Earth and Mars is calculated for each iteration, and then their relative position to the s/c is used to determine the perturbing third-body force according to

(3.4)

where the different vectors are as indicated in Figure 3.1.

<<Figure 3.1>> Relative Position Vectors

Notice that all coordinates use the sun as the origin, rather than the true center of mass for a barycentric analysis. The loss of generality by making this assumption is nominal, especially since Earth and Mars are already assumed to be in circular orbits.

The code checks the veracity of the analysis, by calculation of the Jacobi Integral. This energy integral should be constant for the restricted three-body problem (i.e. for the Hohmann Rendezvous one of the planets must be "turned off" or zeroed, truncating the problem to one of just three bodies). The value of the Jacobi Integral is calculated by

(3.5)

The program is shown to be accurate by zeroing first Earth then Mars, in both cases of which the Jacobi Integral remains constant to eleven significant figures.

NUMERICAL ACCURACY

The Fortran code is compiled and run on a 32-bit PC using double precision arithmetic. The integration is performed by a variable step Runge-Kutta 7/8 routine. The tolerance is set to 1.E-13. The data is calculated using the kgs system of units; kilometers, kilograms, and seconds. The only constants used in the analysis were

Gm for the sun, Earth, and Mars

Semi-major axis for Earth, and Mars

All other constants used in the numerical analysis were derived from these few values, for maximum numerical accuracy.

Overall, the number of floating point operations was kept to an absolute minimum, for accuracy as well as speed of computation: the fewer operations, the less the round off error. In fact, this principal is at the hear of the whole effort to extrapolate the analytical solution as far as possible, leaving as few numerical approximations to be done as necessary. We will now see that the loss of only one or two significant digits can have disproportionate repercussions upon the results. Consequently, the traditional engineering analysis of an interplanetary trajectory, where everything is approximated numerically, is at a disadvantage.

"PATCHED" INTEGRATION

The most vexing problem of the analysis is achieving good accuracy when integrating the s/c motion while inside the Earth's SOI. The ideal solution would have a continuous, seamless integration from launch to landing on Mars. The simulation can, in fact, be set up to do just that using Earth as the central force initially, then the sun. The planets move at a constant angular velocity throughout, with the sun fixed at the origin - what could be simpler?

The source of problems is the difference between the scales by which the motion of the Earth and the s/c are calculated when inside the Earth's SOI. The planet does not move in a continuous motion, but is limited by the trigonometric functions to a certain order of accuracy. Thus, the increment of motion by Earth is proportional to the ratio between the respective distances to the central body. So, when integrating the s/c in near Earth orbit - during which time the RK78 integrator takes extremely small steps to preserve accuracy - the planet Earth "jumps" great distances after many iteration steps, instead of moving at a comparable pace to that of the s/c.

This is not such a drastic assumption as might first be thought. The exact position of the s/c inside Earth's SOI can be used as the starting point in the heliocentric system. Although this suffers the same numerical inaccuracy as before, at least the Jacobi Integral can be used to modify the position. As for the velocity, it is already in absolute coordinates so there is no discrepancy between the two reference frames in this regard. This is a far more accurate transition between the two systems than the abrupt "patched conic" method, and results in an enhanced computational accuracy.

PDE INTEGRATION

An alternate solution is to study the motion of the s/c in equations of motion derived from the 2D Laplace Equation. Instead of the variable step integrator, a grid is set up and a set of partial differential equations is solved numerically. The grid structure is based upon the time step and incremental distances used by the RK78 integrator, thus optimizing the grid spacings in the plane of the orbit. The advantage of this method is that the grid, centered on Earth, moves with the s/c so there is no problem of relative motion. Once the time steps become much greater than the possible increments of motions by Earth, the position and velocity of the s/c are transferred into the geocentric system and the motion is integrated until reaching Earth SOI.

Another distinct advantage of using the Laplace Equation is that the geodesy of Earth can be modeled so that the flight path can be directly integrated from time of launch from the Earth's surface. In this case, the gravitational field of Earth is modeled as a three body problem. The mass of Earth is divided into two increments, a small one which represents the mass in the plane of the orbit, and the rest. The latter is represented by a large mass at the geographic center of Earth, and the small in-plane increment is represented by a small equivalent mass at the appropriate distance from Earth center. A circular orbit by this small mass would represent a uniform in-plane density, while an elliptical orbit can be used to model variations of the planet density at different positions in the orbit.

The benefit of modeling the Earth in this way is that it will be possible to optimize the trajectory with regard to the known geodesy of the Earth.

STRUCTURE

The code is set up with the s/c parameters in a six element state vector such that the following constants apply

position = (x_1,x_2,x_3)

velocity = (v_1,v_2,v_3) (3.6)

Then the perturbations from (3.4) are added to f_1,f_2,f_3 as additional perturbing forces in the x-, y-, and z-directions respectively. The data generated are output to a separate file, then imported into Matlab or Mathematica for further analysis and/or plotting. A full listing of the code is provided in Appendix A, and a flow chart is provided in Figure 3.2.

<<Figure 3.2>> Numerical Approximation

The routine calls one of the two derivative routines, depending on where the s/s is located. If it is within Earth's SOI, then the integration is done as a two body problem, with the Earth as the central body and with no third body perturbations. It is assumed that the sun's influence upon the Earth and the s/c causes both to move with a common mean motion. The earth is already assumed moving at a fixed angular velocity, so the assumption of superposition applies only to the motion of the s/c, and it is necessary to make this assumption to avoid some of the numerical tribulations noted above.

When the s/c reaches Earth's SOI, the program takes the position with respect to Earth and translates it to heliocentric coordinates using the current position of Earth. The heliocentric velocity of the s/c is obtained by adding the mean motion of the Earth to the two body velocity vector. If this translation of coordinates is assumed to take place when Earth is at the x-axis, then the motion of Earth with respect to the sun is in the y-direction only. In so doing, we can assume that the inside-SOI trajectory occurs in the vicinity of the x-axis, at which point the sun is directly behind Earth, which can lead to another helpful approximation, illustrated in Figure 3.3.

<<Figure 3.3>>

As the s/c moves the Earth and s/c are on the same radius from the sun, because the s/c and Earth have close to the same angular velocity (for the +/- two days the orbit remains in the SOI of Earth). This is why it is acceptable to assume superposition of forces within the SOI, that the limits of minimal deviation are less than the usual reckoning of the SOI - so, we have assumed somewhat less than that.