THE INVERSE PROBLEM OF CELESTIAL MECHANICS

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The study of the known positions of the nine planets in our own Solar System begins with the discovery of a new invariant plane. Transforming the data into this new frame of reference leads to an astronomical corollary to the imaginary number "j" and further insights into the very structure of the Solar System. The planets are shown to occupy stability points upon a helical wave in this Ten Body Problem, just as the corners of the equilateral triangle in the Three Body Problem are stable points. This notion is shown to fit the notion of a gravity ellipsoid in the dynamical sense. A Fourier series representation for the planets versus this invariant plane is shown to have potential to develop an analytical basis for the heuristic concepts developed in this primarily conceptual study.

INTRODUCTION

The Inverse Problem of Celestial Mechanics is the study of the positions of the planets in our Solar System. The first such study was done by the ancient astronomer Ptolemy in Alexandria, who assumed a geocentric system and devised an intricate system of epicycles within epicycles to explain the motion of the planets. The next study was done by Johanes Kepler who used the astronomical data gathered by Tycho Brahe to show the planets moved in elliptical orbits about the Sun at one focus, a heliocentric system. From this point forward the Ten Body Problem (sun + nine planets) has been submitted to no further analysis per se. The only exception was an attempt by Poincare to model the motion of the planets using Fourier series expansions, but this was soon abandoned when it was discovered (by Poincare) that Fourier series solutions imply an unstable system.

This report begins with the classical Inverse Problem, looking at the position of the planets in our Solar System with no prior conceptions. In particular, no mass is assumed for the bodies, and no force of gravity between them. Just the positions over time.

THE INVARIANT PLANE

The concept of an invariant plane is that of a special coordinate system about which the motions of the constituent bodies is symmetric. The data for this study are the known positions of the planets. These data are analyzed in a simple numerical method, as follows:

(1) The position of each planet at 0° heliocentric longitude is calculated from the orbital elements.
(2) A linear least squares fit is made of these nine data points, giving a slope and intercept
(3) This is repeated at 10° increments through 360° of longitude.

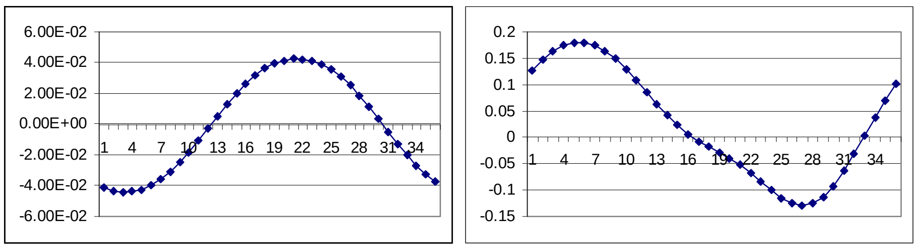

Figure 1 Slope and Intercept data

The slope (Figure 1) of this line varies sinusoidally, which thus defines a flat XY plane in 3D space. The intercept also varies sinusoidally.

The true measure of this new plane in space is how symmetric the motion of the planets is with respect to it. The motion perpendicular to the coordinate system plane is in the Z-direction (Figure 2). All planets (excepting Mercury, Venus, and Pluto) are sinusoids exactly in phase, with the same leading phase angle. Venus and Pluto are exactly pi radians out of phase; they also exhibit retrograde rotation, so being out of phase with the other planets in prograde rotation is a logical consequence. Also, adjusting the intercept to be fixed at the center of the Sun (which itself moves around the barycenter of the Solar System in a small orbit) puts Mercury in phase with the other planets. Again, this exception will be the subject of a separate study.

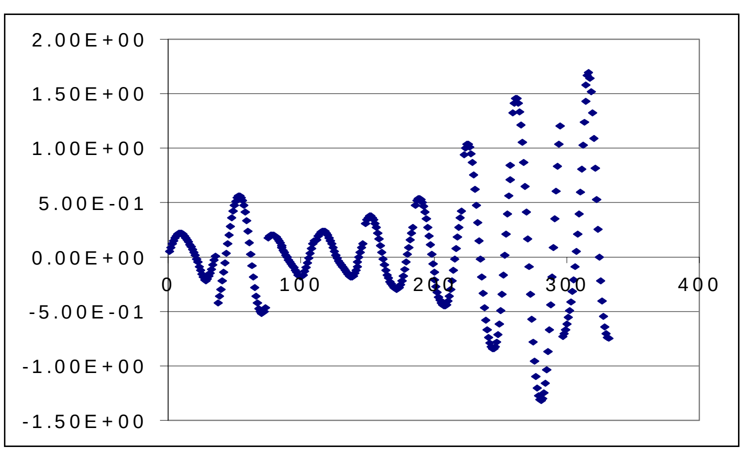

Figure 2 Motion versus the Invariant Plane

THE REGULARIZED TEN BODY PROBLEM

It has been shown that a higher order of symmetry of the Solar System is obtained by transforming into a new coordinate system. This plane is represented by a slope that varies sinusoidally around 360° of heliocentric longitude, and an intercept that also varies sinusoidally around the celestial sphere. These form a 3D plane that is flat with the exception of a distortion at the origin, in the vicinity of the intense gravitational field of the sun. This has been identified as the Invariant Plane.

A straight line can be made to behave in just this manner by considering that a point on a rotating circle traces out a sinusoidal wave form in 2D space. In the current case there are two sinusoidal motions occurring simultaneously, acting upon a straight line. So, attaching one end of a straight line to a circle centered at the origin, and attaching the line (in a sliding link) to another circle at approximately 2 AU, it is possible to cause the line to behave as noted. As both circles rotate, in one complete revolution with 360° of heliocentric longitude, they form the Invariant Plane.

A formal development of the wave generated by this mechanism in 3D space is found in what are known as the Cochoids of Nichomedes which have the parametric equations:

$$x = a + \cos(t) \qquad (1)$$
$$y = a*\tan(t) + \sin(t)$$

Geometrically, these are created by following the motion of one point on the circumference of a circle as it rolls along a flat surface. In 3D the result is a helix. This wave form is able to exhibit two sets of objects 180° out of phase, as required by the conditions of the problem.

The Invariant Plane coordinate transformation has regularized the Ten Body Problem, in a geometrical sense, into a Three Body Problem. Analytically, the parametric equations (1) have done the same thing. Consequently, these equations should form a solution to a Regularized Three Body. The mathematical development in Figures 1 and 2 substitutes equations (1) into the so-called Szebehely Equation for the Regularized Elliptical Three Body Problem. This works because of the geometry of the mechanical model, which determines the value of a in the parametric equations. This, in turn, introduces the complex number "j" into the function which, if accepted as is, leads finally to a closed solution to the Szebehely Equation. The equations (1) regularize the Ten Body Problem into a Three Body Problem.

__Mathematics__

To confirm the data with function (1). First, develop the radius r in the cartesian coordinate system

$$r^2 \quad = x^2 + y^2 \qquad (2)$$

$$- x^2$$

therefore, $r \sim x \sim a + \cos t$ (3)

This can be substituted into the function for y,

$$\begin{aligned} y &= a \tan t + \sin t \\ &= \sin t \,(a/\cos t + 1) \\ &= \tan t \,(a + \cos t) \end{aligned} \qquad (4)$$

Where a = 0.05 in the illustration, but is actually << 0.05. Thus,

$$(a + \cos t) = j \cos t \qquad (5)$$

This will result in rotation about the central point, such that motion is above the system plane. This introduction of the constant "j" is analogous to the development of the orbital model in quantum mechanics. The significance of this assumption, and so the justification for it, will be evident later in the analysis.

$$\begin{aligned} y &= j \sin t \\ x &\cong \cancel{a}^{0} + \cos t \text{ (as before)} = j \cos t \\ x &= j \cos t \quad , \quad \dot{x} = -j \sin t \\ y &= j \sin t \quad , \quad \dot{y} = j \cos t \end{aligned} \qquad (6)$$

Now these values are substituted into the Szebehely equation for the restricted three body problem,

$$\ddot{y} = \frac{(1 + y')^2}{\Lambda^2} \left(\Lambda_y - \dot{y} \Lambda_x \pm 2\sqrt{1 + y'^2} \right) \qquad (7)$$

from which the constant Gamma is defined as

$$\Lambda^2 = \dot{x}^2 + \dot{y}^2 \qquad (8)$$

substituting,

$$\Lambda^2 = -\sin^2 t - \cos^2 t = -1 \quad \therefore \quad \Lambda = j$$
$$\Lambda_x = \Lambda_y = 0$$
$$\text{because} \quad \frac{\partial}{\partial x}(\text{const}) = 0$$

Thus, the Szebehely equation explains the variation from the "system plane wave," defined by the prolate cycloid rotated about the central axis. Substituting back into that equation,

$$-j \sin t = \ddot{y} = \frac{(1 + j\cos t)^2}{-1}\left(\pm 2\sqrt{1 - \cos^2 t}\right) \tag{9}$$

$$-j \sin t = -(1 + j\cos t)^2 (\pm 2 \sin t)$$
$$+j = -(1 + j\cos t)^2 (\pm 2)$$
$$\pm \frac{j}{2} = -1 - 2j\cos t + \cos^2 t$$

Which must satisfy two levels of equality, for the real and the imaginary parts.

(1) $\pm \frac{j}{2} + 2j\cos t = 0$

$$2\cos t = \pm 1/2$$

(2) $1 = \cos^2 t$

$$0 = t$$

where t = 0 is an assumption made for the development of the Szebehely equation.

This fairly reasonable result came because of the one assumption made earlier concerning "j." The dynamics this introduces to the analysis are investigated in the following section, in a heuristic fashion.

The significance of "j" in this analysis is apparently the same as it's conceptualization in engineering dynamics: it implies rotation; here, the rotation is of the body itself, which is contrary to the rotation of the bodies in the adjacent scheme. The complex ordinate about which this rotation occurs is the axis of rotation of the constituent planet(s).

THE SYSTEM WAVE

The first objective of the analysis is to create a more stable system by better explaining systematic deviations from the Invariant Plane. A stable solution set is one of the outcomes of the analysis of the regularized Three Body Problem. Thus, since this approach is from the opposite direction, to create a more stable system is the next logical step toward the mathematical solution sought. This is based on the premise that the more accurately the planetary motions can be defined in this fundamental context, the more stable they should be (ie the less will be their deviation from the norm).

The new plane has already simplified the motion of all the planets. In order to further define the motions, first introduce another variable: the planet’s axis of rotation. This axis is known to always point to the same spot in the celestial sphere, throughout its revolution around the sun. During this movement, the planet moves evenly above and below the ecliptic plane.

These characteristics can be conveniently represented if the planet is considered to be associated with a longitudinal wave. If the axis of rotation is always tied to this sinusoidal wave, then as the planet moves about the sun the elliptical shape of the orbit causes the planet to move closer and farther away from the sun. If this motion happens on a sinusoidal wave, the axis will always point in the same direction. (Figure 1)

Otherwise, the choice of a sinusoidal waveform explains the three movements that occur simultaneously with respect to the new system plane: the elliptical shape of the orbit, the movement above and below the system plane, and the axial inclination of the planets pointing to a fixed spot in the heliocentric sphere while the other motions occur.

Table 1

PlanetInclination of Axis	*X*		
Mercury	5.8 E7	0.0 degrees	1.2 E7 km
Venus	1.1 E8	R179	6.7 E6
Earth	1.5 E8	23.5	2.5 E6
Mars	2.3 E8	25.2	2.1 E7
Jupiter	7.8 E8	3.2	3.8 E7
Saturn	1.4 E9	26.8	7.8 E7
Uranus	2.9 E9	R98	13.2 E7
Neptune	4.5 E9	29.0	3.7 E7
Pluto	5.9 E9	90.0	1.5 E9

The data used in the analysis is listed in Table 1. The semimajor axis is an approximation of the planets’ average orbital distance from the sun. A value "X" is the range of motion of the planet in its orbital plane.

LINEARIZATION

The next task is to find the wavelength of a common "system wave." After trying several solutions, it was evident that the outer planets would be the controlling elements, so the search focused on the lowest common denominator of their orbital range. This value, 23.3 E6 km also had to fit the entire range of movement of the inter planets.

Table 2

Planet	*Division by 23.3 E6*	*Inclination of orbit to the ecliptic*	*Maximum distance from the ecliptic*
Mercury	2 + 1.1 E7 km	7.0 degrees	7.0 E6 km
Venus	4 + 1.7 E7	3.4	6.5 E6
Earth	6 + 1.0 E7	0.0	- -
Mars	9 + 2.0 E7	1.9	7.6 E6
Apollo Group	*9 + 7.8 E6*	-	-
Belt Asteroids	*16 + 2.2 E6*	-	-
Trojan Group	*33 + 3.6 E6*	-	-
Jupiter	33 + 1.1 E6	1.3	17.0 E6
Saturn	60 + 2.0 E6	2.5	61.0 E6
Uranus	124 + 1.0 E7	0.7	35.0 E6
Neptune	193 + 3.1 E6	1.8	141 E6
Pluto	253 + 5.1 E6	17	172 E6

The second column of Table 2 lists the aspect of each planet upon this wave – the remainder of division by the proposed wavelength. This value was used to situate each planet at the proper location upon the wave. Note that positions are only approximate in this two dimensional system: the wave is actually a spiral in three dimensions.

A possible presentation of the "system wave" is a family of curves called the Cochoids of Nichomedes, with the parametric equations:

$$x = a + \cos t \;\&\; y = a \tan t + \sin t \qquad (2)$$

These generate a two-part curve: a loop similar to a prolate cycloid, and a shell. This solution will have some unique possibilities in the analysis of orbits.

The second objective of the analysis was to further develop the study to define the uniform deviations from the new system plane, so that a more stable system could be devised. This has been done for the original positions by simply introducing another variable and trying to match it. By seeking to close fit to this third variable, axial inclination, the affect was to more closely resolve the previously analyzed parameters. This affect becomes more pronounced as the analysis continues in greater detail.

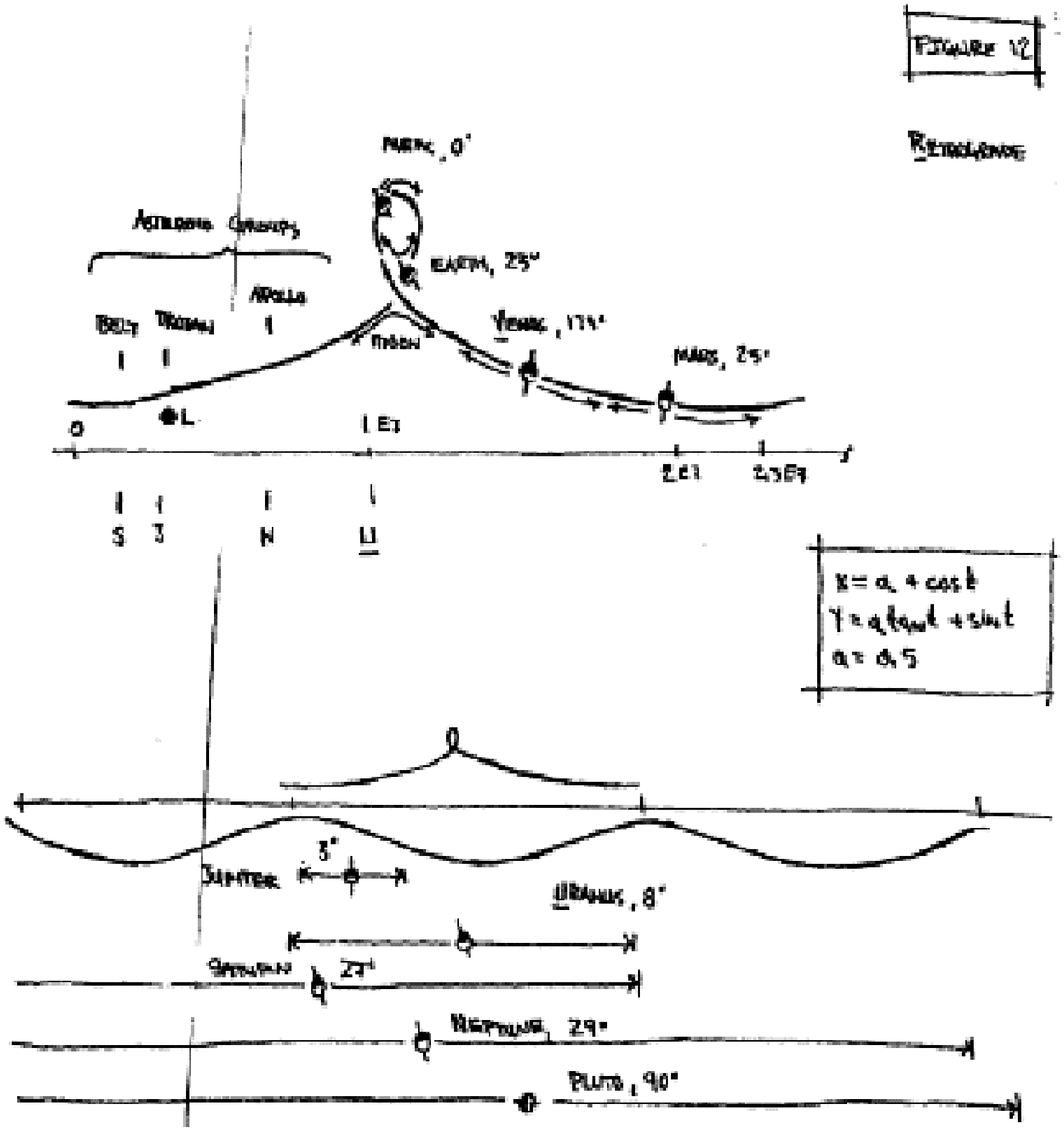

Figure 3 The System Wave

GRAVITY

Figure 3 is a sketch of the planets situated upon either of the two parts of the cycloid, according to the average orbital distances in Table 1. The axis is shown as an arrow pointing according to the right hand rule for the planets' rotation. The positions of the various planets on the waveforms can now be fit to the wave.

First, the motion of Mars moves uniformly on either side of an inflection point in the wave, its axis inclined at about the right amount. Jupiter does the same thing on the lower wave, which arches less to correspond with the lesser axial inclination of Jupiter's axis. Uranus, also on the lower wave, moves through an entire wavelength, with an eight degree axial inclination off the vertical, slightly greater than Jupiter – since it traverses more of the wave.

Next, Saturn on the lower wave traverses two wavelengths, with a much greater axial inclination. The center point of the wave can be thought of as an inflection point, motivating the body to rotate in order to maintain rotational stability – a two dimensional vista of the singularity at the sun discussed earlier. It is interesting to note that all four planets (with the possible exception of Pluto) have almost the same rotational period – about ten hours. This would imply some similarity in the wave analogy, and it is there in the form of the inflection of the wave traveled.

Back to the upper waveform, there is Mercury shown atop the loop, It moves back and forth on the waveform the distance of its ellipticity, maintaining its axis pointed in the same direction. Although the planet is known to have an axial inclination of zero, the wave may be inflected so that the pattern traced is less exaggerated. The geometry of that area of the plane fits this expectation.

Venus is not much farther from the zone where the entire pattern may be warped because of the close coincidence of the two waveforms.

The Earth-Moon system is the most striking aspect of the system, buttoning up the sides of the loop. They are unique in the solar system, the Moon being the largest satellite with respect to its host in the whole solar system. This may be a necessary prerequisite for matter to exist in dynamic stability at the cusp of the wave front as shown. In which case Earth occupies the upper portion of the loop, the moon the lower, in its own orbit. The moon's orbital plane is inclined at five degrees to the Ecliptic, as is indicated by the inflection of the lines (versus the much higher inflection of the upper cusp, indicative of Earth's 23° axial inclination).

Also indicated is a stable quasi-satellite orbit centered on earth, about 80 times the earth-moon distance in cross section. This is about two wavelengths of the system wave, which means the orbit is between two adjacent cusps of the pattern. The stable orbit varies at most 3 to 4 degrees off the ecliptic, which is approximately the extent of the cusp, or small loop in Figure 3. Another such stable orbit exists around Venus, with the same dimensions and aspect. This assumes that the placement of the minor planets on the inner wave is not only unique, but that every portion of the wave must be occupied for overall stability of the solar system. Each planet occupies a special location, which is not shared by any other. In atomic physics terms, the curve defines allowable (stable) orbits.

The Asteroids occupy the area at the beginning of the upper (inner) waveform. The locations of the Belt, Trojan and Apollo Groups of asteroids align closely with the average distances of the planets Saturn, Jupiter and Neptune, respectively. (The Trojan Asteroids are at the Lagrange Point of the Sun-Jupiter system.) This bilateral symmetry is like a zipper, linking the inner planets to the outer planets, and the respective waves by which each is identified.

Another notable phenomenon in the solar system is the "Trojan" orbit. This is a stable orbit that trails a planet. Only one such orbit has been confirmed, that of Eureka, aka Trojan 1990MB. The orbit is 60° behind Mars and inclined at 20° to the orbit of Mars. The nearest cusp on the "system wave" in Figure 3 is $1/6^{th}$ of the distance to the sun, or 60° . Since Mars is shown close to the axis of the wave, a body in the cusp indicated would follow an orbit indicated at the 20° angle to Mars' orbit.

Two more recent astronomical phenomena are worth noting, as further tests of the hypothesis as it presently stands. First, is a marked spiral characteristic recently noted to electromagnetic waves propagating from great distances away, for which there is no known explanation. The presumably gravity-based wave developed thus far has a spiral aspect. Second, the Hale-Bopp comet was seen to have a new type of tail, one that is straight as an arrow and electromagnetic in origin. This takes a far stronger force than a matter based tail. The system pattern traced thus far not only has a straight aspect (ie in the scale of a comet), but this aspect is presumed to be analogous to the strong gravitational force because it has the most affect on the most distant outer planets.

Table 3

Planet	Movement w.r.t. the system plane (Au)	Centerline	Radius (1/2 range)
Mercury	-0.30 + 0.21 = 0.51	-0.05	0.26
Venus	-0.27 + 0.17 = 0.44	-0.05	0.22
Earth	-0.19 + 0.17 = 0.26	-0.06	0.13
Mars	-0.19 + 0.09 = 0.28	-0.05	0.14
Jupiter	-0.10 + 0.02 = 0.12	-0.04	0.06
Saturn	-0.30 + 0.17 = 0.47	-0.06	0.23
Uranus	-0.77 + 0.74 = 1.51	-0.03	0.75
Neptune	-1.21 + 1.21 = 2.42	0.00	1.21
Pluto	-1.10 + 1.60 = 1.70	-0.25	0.85

A final notable characteristic of the analysis is that the midpoint of each planet's motion (and also that of the motion of the y-intercept) is about 0.05 Au below the heliocentric plane calculated by the least squares fit. This fit was normalized to be symmetric about the known origin of the solar system, or the center of the heliocentric sphere. In order for the pronounced symmetry of both systems to be realized, that is: the close symmetry of the least squares fit to a sinusoidal slope and a sinusoidal motion of the y-intercept, and the uniform displacement of the midline of the sinusoidal motion of the planets about this plane, then a two-part pattern such as that envisioned most closely realizes both conclusions. In affect, the defining plane of motion then becomes the bottom half of the above parametric equations wave for the inner planets.

REALITY CHECK

The analysis so far is purposefully not rigorous. The data base itself is an approximation with many factors built in. Thus it would be inaccurate to make anything other than broad generalizations from analysis thereof. The intention of the study so far is to use fairly accurate information to develop a model. It is much easier to fit a model to data that is already a little bit normalized instead of trying to fit a model to the most exacting standards. In this respect the model is similar to the original Bohr model of the atom that worked for only a single atom and electron; nevertheless, it let to important developments thereafter.

The proposed model was shown to explain better than any existing model, several appealing characteristics of the solar system. These are intended to test the model, and to justify further analysis. Had there been no such heuristic proofs, then the model would have been rejected outright. As it is, the model can be considered an "engineering solution" to the situation, because of the following aspects:

The range of the inner planets elliptic orbits fill up the entire wavelength, with insignificant overlapping. The curve therefore represents the only available stable orbits around the sun, as discrete, quantized energy levels.

A newly discovered stable quasi-satellite orbit centered on Earth is indicated, moving between cusps on either side of the earth-moon system, inclined as calculated at a maximum 3° to 4° to the ecliptic.

A similar codependent orbit around Venus, of the same dimensions, matches known calculated values.

A stable node is indicated as a "Trojan" orbit behind the planet Mars, at the correct angular distance and inclination to the orbital plane of Mars.

The position of each planet is closely aligned to its axial inclination, especially considering the presentation is two dimensional and the wave is actually in three.

The outer planets occupy increments of half or full wavelengths, on the lower portion of the waveform. Again, this indicates the only available stable orbits for the outer planets.

The asteroid groups and the average position of the outer planets occupy the same positions, but on the upper and the lower waveforms respectively.

The Trojan asteroids and Jupiter are shown opposite one another, on the two waveforms. The Trojan asteroids are at the stable Lagrange Point of the Sun-Jupiter system.

The planet Mars is located at the junction of the two wave systems, in an area where the asteroids indicate great turbulence. The recent Mars mission showed evidence of huge tidal action by great bodies of water; but with no evidence of water being present in such quantity.

This analysis hints that gravitational wave motion may have caused such geological evidence.

This three-body system exhibits many of the characteristics of the more traditional three-body system that actually has just three point-mass bodies. Most notably is the suggestion of a Lagrange Point, between the Sun and the outer planets being situated at the Trojan asteroids. In additional, various stable orbits are suggested, as might be surmised from the traditional three-body analysis.

This three-body analogy, if true, should be evidenced in a thorough mathematical analysis of the system. As such, the study thus far is a static one: no motion of the bodies is involved, or even implied. The following mathematical solution will introduce a dynamical element.

THE THREE BODY PROBLEM

A proof of the system wave is in the way the Lagrange stability points in the 3BP are found. That is, two circles centered at the primary masses are gradually increased in diameter. The first point where they intersect is L2; the second point L1 (eg the larger mass + the outer zero velocity curve); the third point L3 (eg the smaller mass + the outer zero velocity curve); and L4 and L5 where they finally intersect - both circles + the zero velocity curve. In analytic geometry, intersections of these curves are by definition where the partials are zero; thus they are stability points.

In the following paper it is shown that the significance of "j" is a relative motion between the inner and outer planet regimes. That means the Lagrange Arc from L2 outward to L3/L4 becomes a waveform; similar to the "system wave" derived here. A relative motion between the Lagrange points on the arc, in the non-rotating system, make the arc into a helix. Presumably the arc is twisted into a helix because of a relative motion between the inner and outer planets. The geometry of what is known of the problem can be used to develop an approximate ratio.

The system wave has about 36 wavelengths between the primaries. To make this happen, the relative motion between the inner and outer planet regimes must be 1:36.

This is of the same order as the ratio of the average period of the inner planets (in years: 0.24,0.61,1.0,1.88) versus the outer planets (in years: 11,29,84,164).

So, in engineering terms, the inner gear rotates 36 times for every rotation of the outer gear. If L4 and L5 are attached to seperate gears, then the Lagrange Arc between them will be twisted into a helical waveform such as the "system wave." A synodic period of three years between the two regimes would cause the Earth-moon cusp to rotate once every twelve months, keeping the same face of the moon toward Earth at all times.

This idea of having a synodic period that is the same as the ratio of rotations is the same as for the Earth-moon system. Consequently, our own solar system is at the cusp of a much larger, integalactic "system wave," perhaps with the outer planetary loops paralleling this huge wave to keep our own solar system synchronous with the whole. Just as you would expect, the rotational period of the sun is the same as the Earth's moon - 27 days - making the whole mechanism operate in sync. Also, the moons' gravitational cycle of 18 years coincides with the Saros; an 18 year cycle in which the exact alignment of the Sun, Earth, and Moon repeats.

It is easy to see that this fits in nicely with the mathematical solution. In the following section the veracity of "j" is established. We have just seen that the "system wave" of our solar system interacts with a much large entity on the galactic scale, whereupon the two wave forms sum to 1 (cos^2 + sin^2); thus the so-called Szebehely Equation is solved.

The asteroids identify this theory, as they exist in a continuous range of mean motions near the orbit of Jupiter. If you look at the problem in the rotating 3BP based on the invariant plane, then the effect is that L4 and L5 have relative motion as silhouetted by the asteroids.

An interesting aside is that, in the Ten Body sense our solar system is actually Earth-centered, the great majority of the sun's gravity going into establishing the continuity of our solar system in the Milky Way galaxy and leaving the intricate Earth-moon system to determine the order and cohesiveness of the sun's nine satellites.

THE GRAVITY ELLIPSOID

Another way to look at the development thus far is possible in the concept of an ellipsoid in 3D space. The 3BP orbital plane is a plane though the "gravity ellipsoid" (similar concepts exist in dynamics for angular momentum and inertia), with the L4 and L5 stability points being on the two principal axes, both of which are stable. One of them is the primary axis (with the largest value) and the other is the secondary axis (with the smallest value) which in this case is the one that moves. The third axis is the "system wave," identified in the ellipsoid as a separatrix. This is not an unstable axis, but the boundary between two stability regions surrounding the primary axes - like the seams on a baseball. This is analagous to, in the 3BP orbital plane, to the colinear axis on which the three unstable Lagrange Points L1, L2 and L3 exist but about which objects can be put in a stable orbit (eg the halo orbits around the L2 and L3 points of the Earth-Moon system).

Thus what began as a line representing a continuity of n stable points; ten of which are already identified by the Sun and Planets, has now been expanded into a three dimensional surface; every line on which is a separate continuum of the "Ten Body Problem." In addition, the two stable axes and the unstable "system wave" axis are integral to a "gravity ellipsoid" i.e. a continuum for the "N Body Problem."

FOURIER SERIES

It has previously been shown that an Invariant Plane can be found for the solar system by which the motion of the nine planets is completely symmetrical. That is, a body in an elliptical orbit exhibits sinusoidal behavior in two respects: (1) in the plane of the orbit (this causes the ellipticity), and (2) perpendicular to the plane of the orbit (this causes the inclination of the orbit with respect to the Invariant Plane). This is simply an application of the well known principle of analytical geometry by which it can be shown that an ellipse is the combination of two sinusoidal wave forms.

The exact position of a planet with respect to this Invariant Plane (henceforth the analysis will pertain to a single satellite orbiting a central body) can therefore be represented by the coordinates (in a rotating coordinate system):

$$x = a + b*\sin(phi + c) \qquad (3)$$
$$z = d*\sin(phi + e)$$

Where phi is an angle from some reference point, x is the radial distance from the central body, z is the perpendicular displacement relative to the Invariant Plane, c and e are phase angles. This is simply a mathematical representation of the principles stated above. The first equation for x is the radius vector, and this equations corresponds to the well known expression

$$r = a - ae*\cos E \qquad (4)$$

where E is the eccentric anomaly (one could just as easily use cos in the expressions for x and z; the phase angles c and e make it possible to use either cos or sin). The eccentric anomaly is very nearly the same as the angle phi for orbits of small eccentricity. (Observe that this solution attempts to use a Fourier Series solution to the orbit - a solution practiced 100+ years ago by Poincare' et al but discarded because it implied an unstable dynamic system - thus the term we have here is the first term in a Fourier Series, and need only be approximate because other terms will be added for the exact solution. This is not hard to conceptualize; all orbital mechanics assumes gravity of bodies acts as a single point mass. All this series solution implies is that the body is not of uniform density, and that different aspects of density act individually upon an object in orbit; summing to the actual orbit.)

The correspondence to the normal orbital elements is easily visualizeded in this coordinate system.

(1) eccentricity is the ratio of b/a
(2) inclination is the ratio of d/a
(3) argument of periapse is the angle c
(4) longitude of ascending node is the angle e
(5) eccentric anomaly corresponds to phi
(6) semi major axis corresponds to a

Variations in the orbital elements is an important part of orbital mechanics. Unlike in the conventional coordinate system, these are not abstract concepts. They are rather the result of combining sinusoidal wave forms, as in the interference patterns between gravitational waves. Any shape or orientation of an orbit can be made by the combination of sinusoidal waves. That is the basis of Fourier Analysis. It is how the gravitational field of the Earth is modeled, in fact, as terms in a Fourier Series; making it a straightforward process to correlate aspects of the gravitational model to their precise affect upon the orbit of a satellite.

(1) a sine wave in phase with x changes eccentricity
(2) a sine wave in phase with z changes inclination
(3) a sine wave in phase with c causes precession of the periapse
(4) a sine wave in phase with e causes the ascending node to precess

The coordinate system thus rationalizes the actual force of gravity, and its affect upon physical bodies by virtue of a constructive interference between waves.

Recall that in the graphical derivation of the Invariant Plane in a previous paper, the phase angles for many of the planets were the same. This reduces the number of unknowns in the analysis of a many body problem. The results of that paper - showing all the planets with prograde rotation in phase; all the planets with retrograde rotation in phase; and the two sets of planets pi radians out of phase - also is a proof of this hypothesis, as it shows that all nine planets can be represented by the set of equations stated.

Be that as it may, eliminating unknowns is very important in celestial mechanics. The derivation of the Jacobi Integral in the early days of celestial mechanics resulted in literally tens of thousands of new papers in a field that had been in the doldrums. The above coordinate transformation makes it possible to represent a body in orbit using just five variables; whereas all existing systems must use six. This is a significant development. (The transformation of coordinates into the Invariant Plane eliminates the phase angle for x from both equations if all planets are in prograde rotation or all are in retrograde rotation.)

A common way to reduce the number of variables in problems involving three or more bodies is to assume all the bodies move in the same plane. When studying such a problem in the new coordinate system, two variables are eliminated (d and e); this means that a coplanar 4BP in this coordinate system has only 12 unknowns - the same number of unknowns as the coplanar 3BP in conventional coordinates. This means it is now possible to extend our understanding of the 4BP from a handful of papers into many thousands of papers with relative ease.

FUNCTIONAL ANALYSIS

Fourier Series were used in the late 1800's to model the motion of the planets, by early pioneers in Celestial Mechanics. Then the French mathematician Poincare showed that Fourier Series solutions implied an unstable system, after which time Fourier Series methods were discarded and other methods developed. There has in the last 100 years been no interest in series solutions to planetary motion, and no research done in this area of orbital mechanics.

Fourier Series solutions, however, are quite prevalent in the engineering disciplines. The gravitational field of the Earth is modeled in this way, forming the basis for satellite geodesy. This series model of gravity is used in statistical methods, in the determination of exact satellite orbits. No effort has been made to build a series model of orbits that, when combined with the series model of Earth's gravity field, could generate directly an orbital motion, without any statistical methods.

This report shows the first few terms of a series solution to an elliptical orbit, and suggests how these terms might correlate to the largest term of a planet's gravity model, the J2 term that describes the equatorial bulge. This correlation gives immediate insights into the nature of gravity, as expected from a mathematical versus a statistical method.

ANALYTIC GEOMETRY

A circle is a shape of a uniform radius. Any deviation from this pattern, e.g. an ellipse, can theoretically be constructed by the addition of a Fourier Series to the original circle. An ellipse of small eccentricity can be approximated by fewer terms of a sinusoidal series than an ellipse of high eccentricity.

Consider a unit circle of radius = 1.0 versus an ellipse of eccentricity = .01. Figure 1 shows the change in radius from one focus to the ellipse. The maximum deviation from the unit circle is 0.01

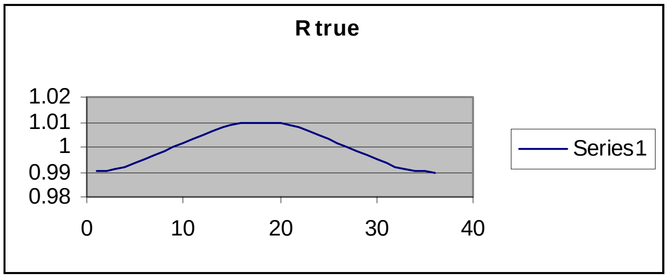

Figure 4 An ellipse of eccentricity .01 versus the unit circle

The first term in a Fourier Series model for this ellipse is to approximate this variation from the unit circle with a simple cosine wave of magnitude .01 Adding this function to the unit circle results in a closer approximation to the true ellipse, and the deviation is shown in Figure 5. Observe that the maximum variation is .00005 and that the difference is similar to two wavelengths of a cosine wave.

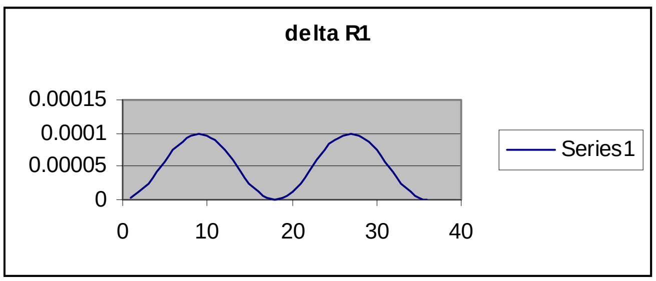

Figure 5 An ellipse versus the first term approximation

The next term in the Fourier Series is approximated by a curve of magnitude .00005 and with have the wavelength of the first term. Forming this function, adding it to the first term approximation, the remainder is shown in Figure 6.

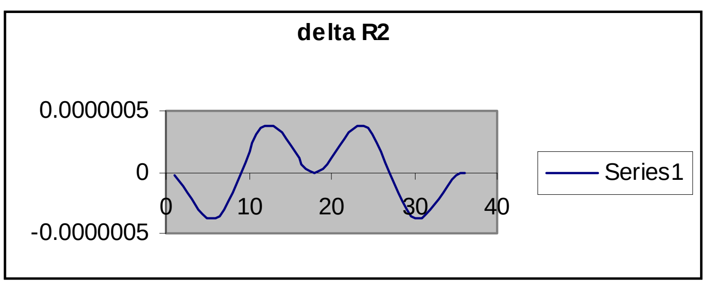

Figure 6 An ellipse versus the second term approximation

The approximation improves by two orders of magnitude with each new term, and the variation remains sinusoidal, but with half the wavelength. This process continues indefinitely, as new terms are added to the Fourier Series approximation.

PHYSICAL THEORY

Fourier Series have been used to model the gravitational field of the Earth and other major celestial bodies, that it is logical to assume some physical significance exists. The basis of such an hypothesis is that ever natural object in a stable elliptical orbit has the same level of dynamic or orbital stability, regardless of the eccentricity of the orbit itself. As eccentricity increases from zero (a circle), some physical aspect of the planet must compensate for the deviation from the ideal.

Each term of the Fourier Series, in such a model, would have some physical significance. The idea being that the orbit created by the first term added to the unit circle is a stable configuration; as is the orbit created by adding the first two terms to the unit circle; and so forth. Not all bodies can exist in these specific energy states, but since they must be in an overall stable state, some physical aspect of the body must enable the body to be stable overall when the body is between the stable orbital states.

Considering the entire dynamic state of a planet, the physical characteristic that has the most obvious affect upon the dynamic stability is the equatorial bulge. This displacement of mass away from the rotational axis gives the body increased rotational stability, which could keep the body in a stable configuration in orbits of non standard eccentricities. If so, then as eccentricity increase beyond a standard value, the magnitude of the equatorial bulge should also increase. Table 4 shows such a correlation between eccentricity and J2, the term in the gravity model corresponding to the equatorial bulge. The major planets are arranged in order of increasing eccentricity.

TABLE 4

Planet	Eccentricity	J2
Venus	.006	.000027
Neptune	.008	.004
Earth	.016	.001
Uranus	.046	.012
Jupiter	.048	.014
Saturn	.055	.016
Mars	.093	.002
Mercury	.205	.00006

The magnitude of J2 in Table 4 increases with eccentricity with the exception of Earth, Mars and Mercury. In the case of Earth, the presence of the Moon which is the largest satellite relative to size of the mother planet in the solar system, the overall J2 in the context of this analysis, would be greater. The other two exceptions, in Mars and Mercury, are more complicated.

The general idea is that the circle is a stable orbit and as the eccentricity of the orbit increases the dynamic stability of the orbit is kept constant by a larger J2 value. The next term of the Fourier Series must be added at some point to make the fit within tolerance to a higher eccentric orbit. This close fit with two terms forms a higher stable orbital, requiring a smaller J2 for stability; after which point the J2 increases as before, for stability. The low J2 for Mars and Mercury indicates the next terms of the Fourier Series come into play at e=0.1 and e=0.2

Coincidentally, observe that all the planets' orbits are in the plane of the ecliptic with the exception of Mars, Venus and Mercury.

Mars	1.85 degrees
Venus	3.4
Mercury	7.0

This inclination may be caused by the distortion in the second term of the Fourier Series, shown in Figure 6. That is, an orbit at a slight inclination with the line of nodes between the two halves of the sinusoid, would balance the two forces laterally and maintain a semblance of equilibrium.

Venus' orbit doesn't fit this scheme. It is nearly circular, and it has almost no equatorial bulge (these fit with the first term approximation); but the orbit has a fairly high inclination to the ecliptic. In addition, Venus exhibits retrograde rotation, rotating on its axis once for every revolution around the Sun. The most likely explanation is that perfectly circular orbits by spherical bodies are unstable (i.e. the body will tend to tumble, rather than rotate on a fixed axis), a situation which is remedied by a highly inclined orbit to the ecliptic, by which forces from the planets in the ecliptic motivate a rotational moment, hence a long term stable orbit.

CONCLUSION : The Unified Field Theory

The initially tentative notion of an "invariant plane" as an integral characteristic of the dynamical system of the planets in our Solar System is shown to be a powerful analytical concept. Transforming the coordinates into this new frame of reference leads to (1) an astronomical analogy of the imaginary number "j", (2) a helical wave upon which the planets inhabit stable points in the Ten Body Problem, (3) a mathematical reduction of this problem of ten bodies to a Three Body Problem, (4) a "gravity ellipsoid" presentation in which the helical wave is the third unstable axis, which is supported by (5) a Fourier series representation of gravity, similar to that used originally by Poincare. All of these important results are substantiated by a wealth of observed astronomical phenomena, which for the first time are shown to have a possible physical significance.

The most important conclusion is that the “water wash” patterns on Mars are caused by the gravitational anomaly that exists between the “invariant plane” and the inertial, mass based plane of reference in the solar system, e.g. the ecliptic or Earth’s orbital plane. This divergence is also proven in the technical paper “Optimizing the Earth to Mars Trajectory,” by means of a numerical model of the region of space in the neighborhood of Mars. This technical paper, as long as the complete Fortran source code used to perform the analysis is available on request. The final section on “Functional Analysis” is intended to develop the equivalent of Taylor Series approximation, but for problems with a spherical symmetry.

Follow up papers show how the Four Body Problem can be used as a model for the semiconductor junction diode – taking advantage of the inverse squared forces that exist in both systems. Thus, electromagnetic and gravitational forces are shown to be equivalent. These two systems are part of a Chaos pattern, by which similar patterns are repeated – like fractals – and scaled intervals. For example, the Schrodinger Electron Cloud Plots are shown to be assimilated from the “System Wave,” thus creating the same complex stable orbitals in the Solar System as exist on the subatomic scale. This is actually simple to show, involving basic geomertric concepts. All together, the complete set of basic forces is shown to exist at all levels. Moreover, since solid state physics models of the semiconductor diode are statistical, there results a real world analogy to these statistical models – a bonus.

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